Foundations of Newtonian Dynamics: An Axiomatic Approach for the Thinking Student

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Abstract. Despite its apparent simplicity, Newtonian Mechanics contains conceptual subtleties that may cause some confusion to the deep-thinking student. These subtleties concern fundamental issues such as, e.g., the number of independent laws needed to formulate the theory, or, the distinction between genuine physical laws and derivative theorems. This article attempts to clarify these issues for the benefit of the student by revisiting the foundations of Newtonian Dynamics and by proposing a rigorous axiomatic approach to the subject. This theoretical scheme is built upon two fundamental postulates, namely, conservation of momentum and superposition property for interactions. Newton's Laws, as well as all familiar theorems of Mechanics, are shown to follow from these basic principles.

1. Introduction

Teaching introductory Mechanics can be a major challenge, especially in a class of students that are not willing to take anything for granted! The problem is that, even some of the most prestigious textbooks on the subject may leave the student with some degree of confusion, which manifests itself in questions like the following:

1. Is Newton's First Law a law of motion (of free bodies) or is it a statement of existence (of inertial reference frames)?
2. Are the first two Newton's Laws independent of each other? It seems that the First Law is but a special case of the Second!
3. Is the Second Law a true law or just a definition (of force)?
4. Is the Third Law more fundamental than conservation of momentum, or is it the other way around?
5. And, finally, how many *independent* laws are really needed in order to build a complete theoretical basis for Mechanics?

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In this article we describe an axiomatic approach to introductory Mechanics that is both rigorous and pedagogical. It purports to clarify issues like the ones mentioned above, at an early stage of the learning process, thus aiding the student to acquire a deep understanding of the basic ideas of the theory. It is not the purpose of this article, of course, to present an outline of a complete course of Mechanics! Rather, we will focus on the most fundamental concepts and principles, those that are taught at the early chapters of Dynamics (we will not be concerned with Kinematics, since this subject confines itself to a description of motion rather than investigating the physical laws governing this motion).

The axiomatic basis of our approach consists of two fundamental postulates, presented in Section 2. The first postulate (*P1*) embodies both the existence of *inertial reference frames* and the *conservation of momentum*, while the second one (*P2*) expresses a *superposition principle for interactions*. The *Law of Inertia* is deduced from *P1*.

In Sec.3, the concept of *force* on a particle subject to interactions is defined (as in *Newton's Second Law*) and *P2* is used to show that a composite interaction of a particle with others is represented by a vector sum of forces. Then, *P1* and *P2* are used to derive the *Law of Action and Reaction*. Finally, a generalization to systems of particles subject to external interactions is made.

For completeness of presentation, certain derivative concepts such as *angular momentum* and *work* are discussed in Sec.4. To make the article self-contained, proofs of all theorems are included.

2. The Fundamental Postulates

We begin with some basic definitions.

Definition 1. A *frame of reference* (or *reference frame*) is a coordinate system (or set of axes) used by an observer to measure the position, orientation, etc., of objects in space. The position of the observer him/herself is assumed *fixed* relative to his/her own frame.

Definition 2. An *isolated system of particles* is a system of particles subject only to their mutual interactions, i.e., subject to no *external* interactions. Any system of particles subject to external interactions that somehow cancel one another in order to make the system's motion identical to that of an isolated system will also be considered an "isolated" system. An isolated system consisting of a single particle is called a *free particle*.

Our first fundamental postulate of Mechanics is stated as follows:

Postulate 1. A class of frames of reference (*inertial frames*) exists such that, for any *isolated* system of particles, a vector equation of the following form is valid:

$$\sum_i m_i \vec{v}_i = \text{constant in time} \qquad (1)$$

where $\vec{v}_i$ is the velocity of the particle indexed by *i* ($i = 1, 2, \cdots$) and m_i is a constant quantity associated with this particle, which quantity is independent of the number or nature of interactions the particle is subject to.

We call m_i the *mass* and $\vec{p}_i = m_i \vec{v}_i$ the *momentum* of this particle. Also, we call

$$\vec{P} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i \qquad (2)$$

the *total momentum* of the system, relative to the considered reference frame. Postulate 1, then, expresses the *principle of conservation of momentum*: the total momentum of an isolated system of particles, relative to an inertial reference frame, is constant in time. (The same is true, in particular, for a free particle.)

Corollary 1. A free particle moves with constant velocity (i.e., with no acceleration) relative to any *inertial* reference frame.

Corollary 2. Any two free particles move with constant velocities relative to each other.

Corollary 3. The position of a free particle may define the origin of an inertial frame of reference.

We note that Corollaries 1 and 2 constitute alternate expressions of the *Law of Inertia* (*Newton's First Law*).

Consider now an isolated system of two particles of masses m_1 and m_2. Assume that the particles are allowed to interact for some time interval *Δt*. By conservation of momentum,

$$\varDelta(\vec{p}_1 + \vec{p}_2) = 0 \;\;\Rightarrow\;\; \varDelta\vec{p}_1 = -\varDelta\vec{p}_2 \;\;\Rightarrow\;\; m_1\,\varDelta\vec{v}_1 = -m_2\,\varDelta\vec{v}_2 \;.$$

We note that the changes in the velocities of the two particles within the (arbitrary) time interval *Δt* must be in opposite directions, a fact that is verified experimentally. Moreover,

$$\frac{|\varDelta\vec{v}_1|}{|\varDelta\vec{v}_2|} = \frac{m_2}{m_1} = \text{constant} \qquad (3)$$

regardless of the kind of interaction or the time *Δt* (which also is an experimentally verified fact). These demonstrate, in practice, the validity of the first postulate. Moreover, Eq.(3) allows us to specify the mass of a particle numerically, relative to the mass of any other particle, by letting the two particles interact for some time.

So far we have examined the case of isolated systems and, in particular, free particles. Consider now a particle subject to interactions with the rest of the world. Then, in general (unless these interactions somehow cancel one another), the particle's momentum will not remain constant relative to an *inertial* reference frame, i.e., will be a function of time. Our second postulate, which expresses the *superposition principle for interactions*, asserts that external interactions act on a particle independently of one another and their effects are superimposed:

Postulate 2. If a particle of mass *m* is subject to interactions with particles $m_1, m_2, \cdots$, then, at each instant *t*, the rate of change of its momentum is equal to

$$\frac{d\vec{p}}{dt} = \sum_i \left(\frac{d\vec{p}}{dt}\right)_i \qquad (4)$$

where $(d\vec{p}/dt)_i$ is the rate of change of the particle's momentum due solely to its interaction with particle m_i (i.e., the rate of change of $\vec{p}$ if the particle *m* interacted only with m_i).

3. The Concept of Force

We now *define* the concept of force, in a manner similar to *Newton's Second Law*:

Definition 3. Consider a particle of mass *m* that is subject to interactions. Let $\vec{p}(t)$ be the particle's momentum as a function of time, as measured relative to an *inertial* reference frame. The vector quantity

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{5}$$

is called the *total force* acting on the particle at time *t*.

Taking into account that, for a single particle, $\vec{p} = m\vec{v}$ with fixed *m*, we may rewrite Eq.(5) in the equivalent form,

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} \tag{6}$$

where $\vec{a}$ is the particle's acceleration at time *t*.

Corollary 4. Consider a particle of mass *m* subject to interactions with particles $m_1, m_2, \cdots$. Let $\vec{F}$ be the total force on *m* at time *t*, and let $\vec{F}_i$ be the force on *m* due solely to its interaction with m_i. Then, by the superposition principle for interactions (Postulate 2) as expressed by Eq.(4), we have:

$$\vec{F} = \sum_i \vec{F}_i \tag{7}$$

Theorem 1. Consider two particles *1* and *2*. Let $\vec{F}_{12}$ be the force on particle *1* due to its interaction with particle *2* at time *t*, and let $\vec{F}_{21}$ be the force on particle *2* due to its interaction with particle *1* at the same instant. Then,

$$\vec{F}_{12} = -\vec{F}_{21} \tag{8}$$

Proof. By the superposition principle, the forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are independent of the presence or not of other particles in interaction with particles *1* and *2*. Thus, without loss of generality, we may assume that the system of the two particles is isolated. Then, by conservation of momentum and by using Eq. (5),

$$\frac{d}{dt}\left(\vec{p}_1+\vec{p}_2\right)=0 \;\Rightarrow\; \frac{d\vec{p}_1}{dt}=-\frac{d\vec{p}_2}{dt} \;\Rightarrow\; \vec{F}_{12}=-\vec{F}_{21}\ .$$

Equation (8) expresses the *Law of Action and Reaction* (*Newton's Third Law*).

Theorem 2. The rate of change of the total momentum $\vec{P}(t)$ of a system of particles, relative to an inertial frame of reference, equals the total *external* force acting on the system at time *t*.

Proof. Consider a system of particles of masses m_i $(i=1,2,\cdots)$. Let $\vec{F}_i$ be the total *external* force on m_i (due to its interactions with particles *not belonging* to the system), and let $\vec{F}_{ij}$ be the *internal* force on m_i due to its interaction with m_j (by convention, $\vec{F}_{ij}=0$ when *i=j*). Then, by Eq.(5) and by taking into account Eq. (7),

$$\frac{d\vec{p}_i}{dt}=\vec{F}_i+\sum_j \vec{F}_{ij}\ .$$

By using Eq.(2) for the total momentum, we have:

$$\frac{d\vec{P}}{dt}=\sum_i \frac{d\vec{p}_i}{dt}=\sum_i \vec{F}_i+\sum_{ij}\vec{F}_{ij}\ .$$

But,

$$\sum_{ij}\vec{F}_{ij}=\sum_{ji}\vec{F}_{ji}=\frac{1}{2}\sum_{ij}\left(\vec{F}_{ij}+\vec{F}_{ji}\right)=0\,,$$

where the action-reaction law (8) has been taken into account. So, finally,

$$\frac{d\vec{P}}{dt}=\sum_i \vec{F}_i=\vec{F}_{ext} \qquad (9)$$

where $\vec{F}_{ext}$ represents the *total external force* on the system.

4. Derivative Concepts and Theorems

Having presented the most fundamental concepts of Mechanics, we now turn to some useful derivative concepts and related theorems, such as those of angular momentum and its relation to torque, work and its relation to kinetic energy, and conservative force fields and their association with mechanical-energy conservation.

Definition 4. Let *O* be the origin of an *inertial* reference frame, and let $\vec{r}$ be the position vector of a particle of mass *m*, relative to *O*. The vector quantity

$$\vec{L}=\vec{r}\times\vec{p}=m\left(\vec{r}\times\vec{v}\right) \qquad (10)$$

(where $\vec{p} = m\vec{v}$ is the particle's momentum in the considered frame) is called the *angular momentum* of the particle relative to *O*.

Theorem 3. The rate of change of the angular momentum of a particle, relative to *O*, is given by

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} \equiv \vec{T} \tag{11}$$

where $\vec{F}$ is the *total* force on the particle at time *t*, and $\vec{T}$ is the *torque* of this force relative to *O*, at this instant.

Proof. Equation (11) is easily proven by differentiating Eq.(10) with respect to time, and by using Eq.(5).

Corollary 5. If the torque of the total force on a particle, relative to some point *O*, vanishes, then the angular momentum of the particle relative to *O* is constant in time (*principle of conservation of angular momentum*).

Under appropriate conditions, the above conservation principle can be extended to the more general case of a system of particles (see, e.g., [1-5]).

Definition 5. Consider a particle of mass *m* in a *force field* $\vec{F}(\vec{r})$, where $\vec{r}$ is the particle's position vector relative to the origin *O* of an inertial reference frame. Let *C* be a curve representing the trajectory of the particle from point *A* to point *B* in this field. Then, the line integral

$$W_{AB} = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} \tag{12}$$

represents the *work* done by the force field on *m* along the path *C*. (*Note:* This definition is valid independently of whether or not additional forces, not related to the field, are acting on the particle; i.e., regardless of whether or not $\vec{F}(\vec{r})$ represents the total force on *m*.)

Theorem 4. Let $\vec{F}(\vec{r})$ represent the *total* force on a particle of mass *m* in a force field. Then, the work done on the particle along a path *C* from *A* to *B* is equal to

$$W_{AB} = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} = E_{k,B} - E_{k,A} = \Delta E_k \tag{13}$$

where

$$E_k = \frac{1}{2} m v^2 = \frac{p^2}{2m} \tag{14}$$

is the *kinetic energy* of the particle.

Proof. By using Eq.(6), we have:

$$\vec{F}\cdot d\vec{r} = m\frac{d\vec{v}}{dt}\cdot d\vec{r} = m\vec{v}\cdot d\vec{v} = \frac{1}{2}m\,d(\vec{v}\cdot\vec{v}) = \frac{1}{2}m\,d(v^2) = m\,v\,dv\,,$$

from which Eq.(13) follows immediately.

Definition 6. A force field $\vec{F}(\vec{r})$ is said to be *conservative* if a scalar function $E_p(\vec{r})$ (*potential energy*) exists, such that the work on a particle along *any* path from *A* to *B* can be written as

$$W_{AB} = \int_A^B \vec{F}(\vec{r})\cdot d\vec{r} = E_{p,A} - E_{p,B} = -\Delta E_p \qquad (15)$$

Theorem 5. If the total force $\vec{F}(\vec{r})$ acting on a particle *m* is conservative, with an associated potential energy $E_p(\vec{r})$, then the quantity

$$E = E_k + E_p = \frac{1}{2}m\,v^2 + E_p(\vec{r}) \qquad (16)$$

(*total mechanical energy* of the particle) remains constant along any path traced by the particle (*conservation of mechanical energy*).

Proof. By combining Eq.(13) (which is generally valid for *any* kind of force) with Eq.(15) (which is valid for *conservative* force fields) we find:

$$\Delta E_k = -\Delta E_p \;\Rightarrow\; \Delta(E_k + E_p) = 0 \;\Rightarrow\; E_k + E_p = const.$$

Theorems 4 and 5 are readily extended to the case of a system of particles [1-5].

5. Summary and Concluding Remarks

Newtonian Mechanics is the first subject in Physics an undergraduate student is exposed to. It continues to be important even at the intermediate and advanced levels, despite the predominant role played there by the more general formulations of Lagrangian and Hamiltonian dynamics.

It is this author's experience as a teacher that, despite its apparent simplicity, Newtonian Mechanics contains certain conceptual subtleties that may leave the deep-thinking student with some degree of confusion. The average student, of course, is happy with the idea that the whole theory is built upon three rather simple laws attributed to Newton's genius. In the mind of the more demanding student, however, puzzling questions often arise, such as, e.g., how many *independent* laws we really need to fully formulate the theory, or, which ones should be regarded as truly fundamental laws of Nature, as opposed to others that can be *derived* as theorems.

This article suggested an axiomatic approach to introductory Mechanics based on two fundamental, empirically verifiable laws, namely, the *principle of conservation of momentum* and the *principle of superposition for interactions*. We showed that all standard ideas of Mechanics (including, of course, Newton's Laws) naturally follow from these basic principles. To make our formulation as economical as possible, we expressed the first principle in terms of a system of particles and treated the single-particle situation as a special case. To make the article self-contained for the benefit of the student, explicit proofs of all theorems were given.

By no means do we assert, of course, that this particular approach is unique or pedagogically superior to other established methods that adopt different viewpoints regarding the axiomatic basis of Classical Mechanics (see, e.g., a historical overview of these viewpoints in the first chapter of [6]). Moreover, this approach suffers from the usual theoretical problems inherent in Newtonian Mechanics (see, e.g., [7,8]), most serious of which is the following: To test whether a given reference frame is inertial or not, one needs to check the constancy or not of the velocity of a free particle, relative to this frame. However, an absolutely "free" particle is only a theoretical conception, for the following reasons: (1) Every particle is subject to the long-range gravitational interaction with the rest of the world. (2) To observe a particle, one necessarily has to somehow interact with it. Thus, no matter how weak this interaction may be, the particle can no longer be considered free during the observation process.

In any case, it looks like Classical Mechanics remains a subject open to discussion and re-interpretation, and more can always be said about things that are usually taken for granted by most students (this is not exclusively their fault, of course!). Happily, some of my own students do not fall into this category. I honestly appreciate the hard time they enjoy giving me in class!

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