On solving Maxwell's equations for a charging capacitor

C. J. Papachristou[1] and A. N. Magoulas[2]

[1]*Department of Physical Sciences, Hellenic Naval Academy, Piraeus 18539, Greece*
E-mail: papachristou@snd.edu.gr

[2]*Department of Electrical Engineering, Hellenic Naval Academy, Piraeus 18539, Greece*
E-mail: aris@snd.edu.gr

Abstract. The charging capacitor is used as a standard paradigm for illustrating the concept of the Maxwell "displacement current". A certain aspect of the problem, however, is often overlooked. It concerns the conditions for satisfaction of the Faraday-Henry law both in the interior and the exterior of the capacitor. In this article the situation is analyzed and a mathematical process is described for obtaining expressions for the electromagnetic field that satisfy the full set of Maxwell's equations both inside and outside the capacitor.

Keywords: Maxwell equations, Faraday-Henry law, displacement current, capacitor

PACS: 03.50.De, 41.20.-q, 01.55.+b, 01.40.Fk

1. Introduction

The charging capacitor is the standard paradigm used in intermediate-level Physics courses, textbooks and articles to demonstrate the significance of the Maxwell "displacement current" (see, e.g., [1-7]). The point is correctly made that, without this "current" term, the static Ampère's law would be incomplete with regard to explaining the conservation of charge as well as the existence of electromagnetic radiation. Also, the line integral of the magnetic field around a closed curve would be an ill-defined concept.

There is, however, a certain subtlety of the situation which is often passed by. It concerns the satisfaction of the Faraday-Henry law both inside and outside the capacitor. Indeed, although care is taken to ensure that the expressions used for the electromagnetic (e/m) field satisfy the Ampère-Maxwell law, no such care is exercised with regard to the Faraday-Henry law. As it turns out, the usual formulas for the e/m field satisfy this latter law only in the special case where the capacitor is being charged at a constant rate. But, if the current responsible for charging the capacitor is time-dependent, this will also be the case with the magnetic field outside the capacitor. This, in turn, implies the existence of an "induced" electric field in that region, contrary to the usual assertion that the electric field outside the capacitor is zero. Moreover, the time dependence of the magnetic field inside the capacitor is not compatible with the assumption that the electric field in that region is uniform, as the case would be in a static situation.

The purpose of this article is to exhibit the theoretical inconsistencies inherent in the "classical" treatment of the charging capacitor problem and to describe a mathematical process

ISSN:1791-4469

for deriving expressions for the e/m field that satisfy the full set of the Maxwell equations (including, of course, the Faraday-Henry law) both inside and outside the capacitor.

After a preliminary discussion of the concept of the electric current through a loop (Section 2), the standard "textbook" approach to the charging-capacitor example in connection with the concept of the displacement current is presented in Section 3. New and more general solutions of the Maxwell system of equations in the interior and the exterior of the capacitor are then derived in Sections 4 and 5, respectively.

2. The current through a loop

Before we proceed to write the Ampère-Maxwell law in its integral form, we must carefully define the concept of the *total current through a loop C* (where by "loop" we mean a closed curve in space).

Proposition. Consider a region R of space within which the distribution of charge, expressed by the volume charge density ρ, is time-independent ($\partial\rho/\partial t=0$). Let C be an oriented loop in R, and let S be any open surface in R bordered by C and oriented accordingly. We define the total current through C as the surface integral of the current density $\vec{J}$ over S:

$$I_{in} = \int_S \vec{J} \cdot \overrightarrow{da} \tag{1}$$

Then, the quantity I_{in} has a well-defined value independent of the particular choice of S (that is, I_{in} is the same for all open surfaces S bounded by C).

Proof. By the equation of continuity for the electric charge (see, e.g., [8], Chap. 6) and by the fact that the charge density ρ inside the region R is static ($\partial\rho/\partial t=0$), we have that $\vec{\nabla} \cdot \vec{J} = 0$. Therefore, within this region of space the current density has the properties of a solenoidal field. In particular, the value of the surface integral of $\vec{J}$ will be the same for all open surfaces S sharing a common border C.

As an example, let us consider a circuit carrying a time-dependent current $I(t)$. If the circuit does not contain a capacitor, no charge is piling up at any point and the charge density at any elementary segment of the circuit is constant in time. Moreover, at each instant t, the current I is constant along the circuit, its value changing only with time. Now, if C is a loop encircling some section the circuit, as shown in Fig. 1, then, at each instant t, the same current $I(t)$ will pass through any open surface S bordered by C. Thus, the integral in (1) is well defined for all t, assuming the same value $I_{in}=I(t)$ for all S.

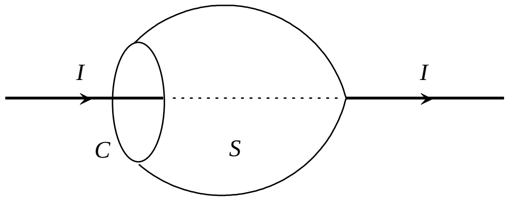

Figure 1

Things change if the circuit contains a capacitor which is charging or discharging. It is then no longer true that the current $I(t)$ is constant along the circuit; indeed, $I(t)$ is zero inside the capacitor and nonzero outside. Thus, the value of the integral in (1) depends on whether the surface S does or does not contain points belonging to the interior of the capacitor.

3. Maxwell displacement current in a charging capacitor

Figure 2 shows a simple circuit containing a capacitor that is being charged by a time-dependent current $I(t)$. At time t, the plates of the capacitor, each of area A, carry charges $\pm Q(t)$.

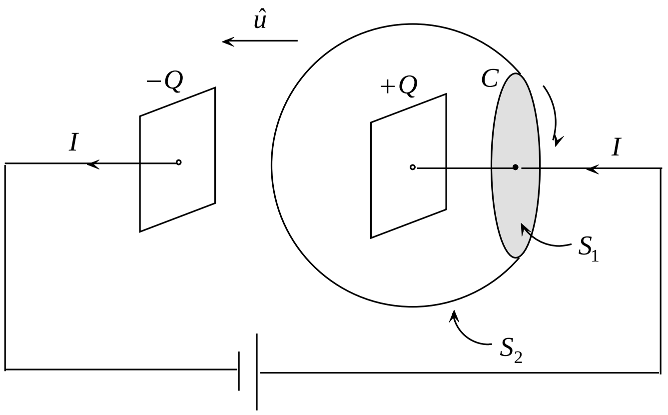

Figure 2

Assume that we encircle the current I by an imaginary plane loop C parallel to the positive plate and oriented in accordance with the "right-hand rule", consistently with the direction of I (this direction is indicated by the unit vector $\hat{u}$). The "current through C" is here an ill-defined notion since the value of the integral in Eq. (1) is $I_{in}=I$ for the flat surface S_1 and $I_{in}=0$ for the curved surface S_2 (Fig. 2). This, in turn, implies that Ampère's law of magnetostatics [1-4,8] cannot be valid in this case, given that, according to this law, the integral of the magnetic field $\vec{B}$ along the loop C, equal to $\mu_0 I_{in}$, would not be uniquely defined but would depend on the surface S bounded by C.

Maxwell restored the single-valuedness of the closed line integral of $\vec{B}$ by introducing the so-called *displacement current*, which is essentially the rate of change of a time-dependent electric field:

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \Leftrightarrow \quad I_d = \int_S \vec{J}_d \cdot \overrightarrow{da} = \varepsilon_0 \int_S \frac{\partial \vec{E}}{\partial t} \cdot \overrightarrow{da} \qquad (2)$$

The *Ampère-Maxwell law* reads:

ISSN:1791-4469

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\quad\Leftrightarrow$$
$$\oint_C\vec{B}\cdot\vec{dl}=\mu_0 I_{in}+\varepsilon_0\mu_0\int_S\frac{\partial\vec{E}}{\partial t}\cdot\vec{da}\equiv\mu_0(I+I_d)_{in}\qquad(3)$$

where I_{in} is given by Eq. (1).

Now, the standard “textbook” approach to the charging capacitor problem goes as follows: Outside the capacitor the electric field vanishes everywhere, while inside the capacitor the electric field is uniform – albeit time-dependent – and has the static-field-like form

$$\vec{E}=\frac{\sigma(t)}{\varepsilon_0}\hat{u}=\frac{Q(t)}{\varepsilon_0 A}\hat{u}\qquad(4)$$

where $\sigma(t)=Q(t)/A$ is the surface charge density on the positive plate at time t. This density is related to the current I, which charges the capacitor, by

$$\sigma'(t)=\frac{Q'(t)}{A}=\frac{I(t)}{A}\qquad(5)$$

(the prime indicates differentiation with respect to t). Thus, inside the capacitor,

$$\frac{\partial\vec{E}}{\partial t}=\frac{\sigma'(t)}{\varepsilon_0}\hat{u}=\frac{I(t)}{\varepsilon_0 A}\hat{u}\qquad(6)$$

Outside the capacitor the time derivative of the electric field vanishes everywhere and, therefore, so does the displacement current.

Now, on the flat surface S_1 the total current through C is $(I+I_d)_{in}=I+0=I(t)$. The Ampère-Maxwell law (3) then yields:

$$\int_C\vec{B}\cdot\vec{dl}=\mu_0 I(t)\qquad(7)$$

On the curved surface S_2, the total current through C is $(I+I_d)_{in}=0+I_{d,in}=I_{d,in}$, where the quantity on the right assumes a nonzero value only for the portion S_2' of S_2 which lies inside the capacitor. This quantity is equal to

$$I_{d,in}=\varepsilon_0\int_{S_2'}\frac{\partial\vec{E}}{\partial t}\cdot\vec{da}=\frac{I(t)}{A}\int_{S_2'}\hat{u}\cdot\vec{da}\qquad(8)$$

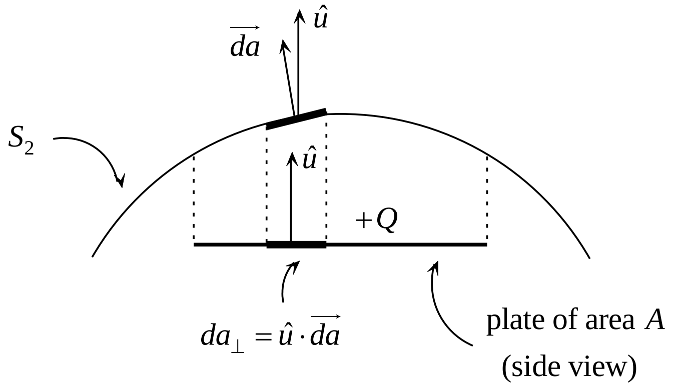

Figure 3

The dot product in the integral on the right of (8) represents the projection of the surface element $\overrightarrow{da}$ onto the axis defined by the unit vector $\hat{u}$ (see Fig. 3). This is equal to the projection $da_\perp$ of an elementary area da of $S_2{}'$ onto the flat surface of the plate of the capacitor. Eventually, the integral on the right of (8) equals the total area A of the plate. Hence, $I_{d,in}=I(t)$ and, given that $I_{in}=0$ on S_2, the Ampère-Maxwell law (3) again yields the result (7).

So, everything works fine with regard to the Ampère-Maxwell law, but there is one law we have forgotten so far; namely, the *Faraday-Henry law*! According to that law, a time-changing magnetic field is always accompanied by an electric field (or, as is often said, "induces" an electric field). So, the electric field outside the capacitor cannot be zero, as claimed previously, given that the time-dependent current $I(t)$ is expected to generate a time-dependent magnetic field. For a similar reason, the electric field inside the capacitor cannot have the static-field-like form (4) (there must also be a contribution from the rate of change of the magnetic field between the plates).

An exception occurs if the current I which charges the capacitor is constant in time, since in this case the magnetic field will be static everywhere. This is actually the assumption silently or explicitly made in many textbooks (see, e.g., [2], Chap. 21). Physically this means that the capacitor is being charged at a constant rate. But, in the general case where $I(t)\neq$constant, the preceding discussion regarding the charging capacitor problem needs to be significantly revised in order to take into account the entire set of the Maxwell equations; in particular, the Ampère-Maxwell law as well as the Faraday-Henry law.

4. The Maxwell equations inside the capacitor

We consider a parallel-plate capacitor with circular plates of radius a, thus of area $A=\pi a^2$. The space in between the plates is assumed to be empty of matter. The capacitor is being charged by a time-dependent current $I(t)$ flowing in the $+z$ direction. The z-axis is perpendicular to the plates (the latter are therefore parallel to the xy-plane) and passes through their centers, as seen in Fig. 4 (by $\hat{u}_z$ we denote the unit vector in the $+z$ direction).

ISSN:1791-4469

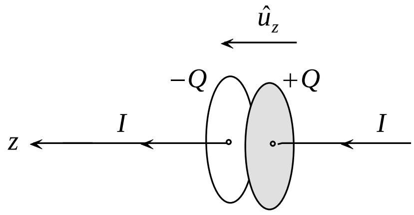

Figure 4

The capacitor is being charged at a rate $dQ/dt=I(t)$, where $+Q(t)$ is the charge on the right plate (as seen in the figure) at time t. If $\sigma(t)=Q(t)/\pi a^2=Q(t)/A$ is the surface charge density on the right plate, then the time derivative of σ is given by Eq. (5).

We assume that the plate separation is very small compared to the radius a, so that the electromagnetic (e/m) field inside the capacitor is practically independent of z, although it *does* depend on the normal distance ρ from the z-axis. (We will not be concerned with edge effects, thus we will restrict out attention to points that are not too close to the edges of the plates.) In cylindrical coordinates (ρ, φ, z) the magnitude of the e/m field at any time t will thus only depend on ρ (it will not depend on the angle φ, as follows by the symmetry of the problem).

We assume that the positive and the negative plate of the capacitor of Fig. 4 are centered at z=0 and z=d, respectively, on the z-axis, where, as mentioned above, the plate separation d is much smaller than the radius a of the plates. The interior of the capacitor is then the region of space with $0 \le \rho < a$ and $0 < z < d$.

The magnetic field inside the capacitor is azimuthal, of the form $\vec{B} = B(\rho, t)\hat{u}_\varphi$. As noted in Sec. 3, a standard practice is to assume that, at all t, the electric field in this region is uniform, of the form

$$\vec{E} = \frac{\sigma(t)}{\varepsilon_0}\,\hat{u}_z \tag{9}$$

while everywhere outside the capacitor the electric field vanishes. With this assumption the magnetic field inside the capacitor is found to be [2,3,6]

$$\vec{B} = \frac{\mu_0 I(t)\rho}{2\pi a^2}\,\hat{u}_\varphi = \frac{\mu_0 I(t)\rho}{2A}\,\hat{u}_\varphi \tag{10}$$

Expressions (9) and (10) must, of course, satisfy the Maxwell system of equations in empty space, which system we write in the form [1,8]

$$\begin{array}{llll} (a) & \vec{\nabla}\cdot\vec{E} = 0 & (c) & \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ (b) & \vec{\nabla}\cdot\vec{B} = 0 & (d) & \vec{\nabla}\times\vec{B} = \varepsilon_0\mu_0\dfrac{\partial\vec{E}}{\partial t} \end{array} \tag{11}$$

By using cylindrical coordinates (see Appendix) and by taking into account that $\sigma'(t)=I(t)/A$ [Eq. (5)], it is not hard to show that (9) and (10) satisfy three of Eqs. (11), namely, (*a*), (*b*) and (*d*). This is not the case with the Faraday-Henry law (11*c*), however, since by (9) and (10) we find that $\vec{\nabla}\times\vec{E}=0$, while

$$\frac{\partial\vec{B}}{\partial t}=\frac{\mu_0 I'(t)\rho}{2A}\,\hat{u}_\varphi \ .$$

An exception occurs if the current I is constant in time, i.e., if the capacitor is being charged at a constant rate, so that $I'(t)=0$. But, for a current $I(t)$ with arbitrary time dependence, the pair of fields (9) and (10) does not satisfy the third Maxwell equation.

To remedy the situation and restore the validity of the full set of Maxwell's equations in the interior of the capacitor, we must somehow correct the expressions (9) and (10) for the e/m field. To this end, we employ the following *Ansatz*:

$$\vec{E}=\left(\frac{\sigma(t)}{\varepsilon_0}+f(\rho,t)\right)\hat{u}_z \ , \quad \vec{B}=\left(\frac{\mu_0 I(t)\rho}{2A}+g(\rho,t)\right)\hat{u}_\varphi \ ; \qquad (12)$$
$$\sigma'(t)=I(t)/A$$

where $f(\rho,t)$ and $g(\rho,t)$ are functions to be determined consistently with the given current function $I(t)$ and the given initial conditions. It is easy to check that the solutions (12) automatically satisfy the first two Maxwell equations (11*a*) and (11*b*). By the Faraday-Henry law (11*c*) and the Ampère-Maxwell law (11*d*) we get the following system of partial differential equations:

$$\frac{\partial f}{\partial\rho}=\frac{\partial g}{\partial t}+\frac{\mu_0 I'(t)\rho}{2A} \qquad (a)$$
$$\frac{\partial(\rho g)}{\partial\rho}=\varepsilon_0\mu_0\frac{\partial(\rho f)}{\partial t} \qquad (b) \qquad (13)$$

Note in particular that the "classical" solution with $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$ is possible only if $I'(t)=0 \Leftrightarrow I=$constant in time (i.e., if the capacitor is being charged at a constant rate), as mentioned earlier.

As a special case, let us assume that the functions f and g are time-independent, i.e., $\partial f/\partial t = \partial g/\partial t=0 \Leftrightarrow f=f(\rho),\ g=g(\rho)$. From (13*a*) we get (ignoring an arbitrary constant):

$$f(\rho)=\frac{\mu_0 I'(t)\rho^2}{4A} \ .$$

This can only be valid if $I'(t)=$constant $\Leftrightarrow I''(t)=0$. On the other hand, (13*b*) yields: $\rho g=$constant $\equiv\lambda \Leftrightarrow g(\rho)=\lambda/\rho$. In order for $g(\rho)$ to be finite for $\rho=0$, we must set $\lambda=0$, so that $g(\rho)\equiv 0$. The solution (12) for the e/m field inside the capacitor is then written:

$$\vec{E} = \left(\frac{\sigma(t)}{\varepsilon_0} + \frac{\mu_0 I'(t)\rho^2}{4A} \right) \hat{u}_z \ , \quad \vec{B} = \frac{\mu_0 I(t)\rho}{2A} \hat{u}_\varphi \ ; \qquad (14)$$
$$I''(t) = 0 \ , \quad \sigma'(t) = I(t)/A$$

We notice that, since $I''(t)=0$, Eq. (6) is still valid and the displacement current inside the capacitor is again given by $I_d = I(t)$. What *is* different here is the correction to the electric field in order for the Faraday-Henry law to be satisfied.

5. The Maxwell equations outside the capacitor

We recall that the positive and the negative plate of the capacitor of Fig. 4 are centered at $z=0$ and $z=d$, respectively, on the z-axis, where the plate separation d is much smaller than the radius a of the plates. The space exterior to the capacitor consists of points with $\rho > 0$ and $z \notin (0,d)$, as well as points with $\rho > a$ and $0 < z < d$. (In the former case we exclude points on the z-axis, with $\rho=0$, to ensure the finiteness of our solutions in that region.)

The e/m field outside the capacitor is usually described mathematically by the equations [2,3,6]

$$\vec{E} = 0 \ , \quad \vec{B} = \frac{\mu_0 I(t)}{2\pi\rho} \hat{u}_\varphi \qquad (15)$$

As the case is with the standard solutions in the interior of the capacitor, the solutions (15) fail to satisfy the Faraday-Henry law (11c) (although they do satisfy the remaining three Maxwell equations), since $\vec{\nabla} \times \vec{E} = 0$ while

$$\frac{\partial \vec{B}}{\partial t} = \frac{\mu_0 I'(t)}{2\pi\rho} \hat{u}_\varphi \ .$$

As before, an exception occurs if the current I is constant in time, i.e., if the capacitor is being charged at a constant rate, so that $I'(t)=0$.

To find more general solutions that satisfy the entire set of the Maxwell equations, we work as in the previous section. Thus, we assume the following general form of the e/m field everywhere outside the capacitor:

$$\vec{E} = f(\rho,t)\, \hat{u}_z \ , \quad \vec{B} = \left(\frac{\mu_0 I(t)}{2\pi\rho} + g(\rho,t) \right) \hat{u}_\varphi \qquad (16)$$

where f and g are functions to be determined consistently with the given current function $I(t)$. The solutions (16) automatically satisfy the first two Maxwell equations (11a) and (11b). By Eqs. (11c) and (11d) we get the following system of partial differential equations:

$$\frac{\partial f}{\partial \rho} = \frac{\partial g}{\partial t} + \frac{\mu_0 I'(t)}{2\pi\rho} \qquad (a)$$
$$\frac{\partial(\rho g)}{\partial \rho} = \varepsilon_0 \mu_0 \frac{\partial(\rho f)}{\partial t} \qquad (b) \qquad (17)$$

Again, the usual solution with $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$ is possible only if $I'(t)=0$, i.e., if the capacitor is being charged at a constant rate.

As a special case, let us assume that the functions f and g are time-independent, i.e., $f=f(\rho)$, $g=g(\rho)$. From (17*a*) we get:

$$f(\rho) = \frac{\mu_0 I'(t)}{2\pi} \ln(\kappa\rho)$$

where κ is a positive constant quantity having dimensions of inverse length. This can only be valid if $I'(t)$=constant $\Leftrightarrow I''(t)=0$. On the other hand, (17*b*) yields: ρg=constant $\equiv \lambda \Leftrightarrow g(\rho)=\lambda/\rho$. Since $\rho>0$, by assumption, we could now let $\lambda\neq 0$. For reasons of continuity, however (see below), we set $\lambda=0$, so that $g=0$. The solution (16) for the e/m field outside the capacitor is then written:

$$\vec{E} = \frac{\mu_0 I'(t)}{2\pi} \ln(\kappa\rho)\, \hat{u}_z \,, \quad \vec{B} = \frac{\mu_0 I(t)}{2\pi\rho}\, \hat{u}_\varphi \,; \qquad (18)$$
$$I''(t) = 0$$

Note, in particular, that the magnetic field in the strip $0<z<d$ is continuous for $\rho=a$, since the expression for $\vec{B}$ in (18) matches the corresponding expression in (14) upon substituting $\rho=a$ (remember that $A=\pi a^2$). No analogous continuity exists, however, for the electric field. Physically, this may be attributed to fringing effects at the edges of the plates.

6. Summary

The purpose of this article is to point out the need to revisit the problem of the charging capacitor, as this is discussed in connection with the Maxwell displacement current, and to carefully examine the expressions for the e/m field both in the interior and the exterior of this system. As was noted, the standard formulas assumed for this field, tailor-made to satisfy the Ampère-Maxwell law, fail to satisfy the Faraday-Henry law except in the special case where the capacitor is being charged at a constant rate. We have derived general expressions for the e/m field that satisfy the full set of Maxwell's equations for arbitrary charging rate of the system. These results may reduce to the familiar set of equations in the case of a constant charging rate.

Note

This article is an extensively revised and expanded version of an article published previously in letter form [9]. In particular, the results contained in Sec. 5 of this article are new.

ISSN:1791-4469

Appendix: Vector operators in cylindrical coordinates

Let $\vec{A}$ be a vector field, expressed in cylindrical coordinates (ρ, φ, z) as

$$\vec{A} = A_\rho(\rho,\varphi,z)\,\hat{u}_\rho + A_\varphi(\rho,\varphi,z)\,\hat{u}_\varphi + A_z(\rho,\varphi,z)\,\hat{u}_z \ .$$

The *div* and the *rot* of this field, in this system of coordinates, are written respectively as follows:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z} \ ,$$

$$\vec{\nabla}\times\vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{u}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{u}_\varphi + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\varphi) - \frac{\partial A_\rho}{\partial\varphi}\right)\hat{u}_z \ .$$

In particular, if the vector field is of the form $\vec{A} = A_\varphi(\rho)\,\hat{u}_\varphi + A_z(\rho)\,\hat{u}_z$, then $\vec{\nabla}\cdot\vec{A} = 0$.

References

1. D. J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition (Prentice-Hall, 1999).
2. R. K. Wangsness, *Electromagnetic Fields*, 2nd Edition (Wiley, 1986).
3. A. Shadowitz, *The Electromagnetic Field* (McGraw-Hill, 1975).
4. V. Rojansky, *Electromagnetic Fields and Waves* (Dover, 1979).
5. J. D. Jackson, *Maxwell's Displacement Current Revisited*, Eur. J. Phys. 20 (1999) 495.
6. K. T. McDonald, *Magnetic Field in a Time-Dependent Capacitor* (2017), http://www.physics.princeton.edu/~mcdonald/examples/displacement.pdf
7. K. T. Selvan, *A Revisiting of Scientific and Philosophical Perspectives on Maxwell's Displacement Current*, IEEE Antennas and Propagation Magazine, Vol. 51, No. 3 (2009) 36.
8. C. J. Papachristou, *Introduction to Electromagnetic Theory and the Physics of Conducting Solids* (META Publishing, Nov. 2017[1]); also, arXiv:1711.09969[2].
9. C. J. Papachristou, *Some Remarks on the Charging Capacitor Problem*, Advanced Electromagnetics, Vol. 7, No. 2 (2018), pp. 10-12.
https://www.aemjournal.org/index.php/AEM/article/view/694

[1] http://metapublishing.org/index.php/MP/catalog/book/52

[2] https://arxiv.org/abs/1711.09969